Proceedings of the Workshop on Applied Topological Structures

editors
Jesús Rodríguez-López and Pedro Tirado

WATS'15

Valencia, Spain, September 3-4, 2015

Universitat Politècnica de València

EDITORS

Jesús Rodríguez-López and Pedro Tirado

Proceedings of the Workshop on Applied Topological Structures

WATS'15

Editorial Universitat Politècnica de València

Proceedings of the Workshop on Applied Topological Structures WATS'15

Editors: Jesús Rodríguez-López
Pedro Tirado

www.lalibreria.upv.es / Ref.: 6264_01_01_01

ISBN: 978-84-9048-412-8 (printed version)

Workshop on Applied Topological Structures WATS'15

September 3-4, 2015

Valencia, Spain

Scientific Committee

Iraide Mardones *(Universidad del País Vasco, Spain)*

Jesús Rodríguez-López *(Universitat Politècnica de València, Spain)*

Manuel Sanchis *(Universitat Jaume I, Castelló, Spain)*

Almanzor Sapena *(Universitat Politècnica de València, Spain)*

Organizing Committee

Pedro Tirado - *Chairman (Universitat Politècnica de València, Spain)*

Carmen Alegre *(Universitat Politècnica de València, Spain)*

Valentín Gregori *(Universitat Politècnica de València, Spain)*

Josefa Marín *(Universitat Politècnica de València, Spain)*

Sponsors and Collaborating Institutions

Ministerio de Economía y Competitividad

Instituto Universitario de Matemática Pura y Aplicada

Project MTM2012-37894-C02-01

Contents

Preface

General Topology has become one of the fundamental parts of mathematics. Nowadays, as a consequence of an intensive research activity, this mathematical branch has been shown to be very useful in modeling several problems which arise in some branches of applied sciences as Economics, Artificial Intelligence and Computer Science. Due to this increasing interaction between applied and topological problems, we have promoted the creation of an annual or biennial workshop to encourage the collaboration between different national and international research groups in the area of General Topology and its Applications. This year it has been given the name of Worksop on Applied Topological Structures (WATS).

This book contains a collection of papers presented by the participants in this workshop which took place in Valencia (Spain) from September 3 to 4, 2015.

All the papers of the book have been strictly refereed.

We would like to thank all participants, the plenary speakers and the regular ones, for their excellent contributions.

We express our gratitude to the Ministerio de Economía y Competitividad, grant MTM2012-37894-C02-01, and Instituto de Matemática Pura y Aplicada for their financial support without which this workshop would not have been possible.

We are certain of all participants have established fruitful scientific relations during the Workshop.

The Organizing Committee of WATS'15

List of participants

Carmen Alegre *(Universitat Politècnica de València, Spain)*

Francisco Castro-Company *(Gilmation SL, Spain)*

Juan José Font *(Universitat Jaume I, Spain)*

José F. Gálvez-Rodríguez *(Universidad de Almería, Spain)*

Valentín Gregori *(Universitat Politècnica de València, Spain)*

Javier Gutiérrez García *(Universidad del País Vasco, Spain)*

Iraide Mardones *(Universidad del País Vasco, Spain)*

Josefa Marín *(Universitat Politècnica de València, Spain)*

Juan José Miñana *(Universitat Politècnica de València, Spain)*

Jesús Rodríguez-López *(Universitat Politècnica de València, Spain)*

Salvador Romaguera *(Universitat Politècnica de València, Spain)*

Miguel Ángel Sánchez-Granero *(Universidad de Almería, Spain)*

Manuel Sanchis *(Universitat Jaume I, Spain)*

Almanzor Sapena *(Universitat Politècnica de València, Spain)*

Pedro Tirado *(Universitat Politècnica de València, Spain)*

WATS'15

LECTURES

Workshop on Applied Topological Structures
Valencia, 3-4 September 2015
pp. 9 – 16

On fixed point theorems for generalized set valued maps with mw-distances

Carmen Alegre and Josefa Marín[1]

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, 46022 Valencia, Spain

(calegre@mat.upv.es, jomarinm@mat.upv.es)

Abstract

In this paper the notion of mw-distance on a quasi-metric space is discussed. Some fixed point theorems in the context of quasi-metric spaces using that notion are included.

Key words: *quasi-metric space, complete quasi-metric space, fixed point, set-valued map, generalized contraction, mw-distance.*

1. Introduction and preliminaries

Kada et al. [11] introduced the notion of w-distance on a metric space and improved some classical fixed point theorems by replacing the metric with a w-distance in the contraction conditions. Later, Park [19] extended this notion of

[1]The authors acknowledge the support of the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

w-distance to quasi-metric spaces. Using these concepts, Alegre, Marín and Romaguera ([4], [5]) obtained fixed point theorems for generalized contractions on complete metric and quasi-metric spaces which involve w-distances and Meir-Keeler or Jachymski functions.

A metric d on X is a w-distance on the metric space (X, d). Nevertheless, if d is a quasi-metric on X, d is not necessarily a $w-$distance on the quasi-metric space (X, d). Motivated from this fact, we introduced in [2] the notion of mw-distance on a quasi-metric space, slightly modifying the definition of w-distance given by Park. This new notion generalizes the concept of quasi-metric. We also showed that mw-distance and w-distance are two different notions, both in the metric case and quasi-metric case.

By using mw-distances, it has been possible to obtain new fixed point theorems for generalized contractions on quasi-metric spaces [2] and generalizations of well-known fixed points theorems in metric spaces (see [1], [16]). Currently, our purpose is to obtain fixed point theorems for multivalued maps on quasi-metric spaces with mw-distances.

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of real numbers, the set of non-negative real numbers and the set of positive integer numbers, respectively. Our basic references for quasi-metric spaces and asymmetric normed spaces are [9], [13] and [7].

A quasi-metric on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If in addition it is fulfilled (iii) $d(x, y) = d(y, x)$ for all $x, y \in X$, d is a metric on X.

If the quasi-metric d satisfies the stronger condition (i) $d(x, y) = 0$ if and only if $x = y$, we say that d is a T_1 quasi-metric on X.

A (T_1) quasi-metric space is a pair (X, d) such that X is a non-empty set and d is a (T_1) quasi-metric on X.

Each quasi-metric d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X,\ \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Note that if d is quasi-metric then τ_d is a T_0 topology, and if d is a T_1 quasi-metric then τ_d is a T_1 topology on X.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$, is also a quasi-metric on X, and the function d^s defined by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ for all $x, y \in X$, is a metric on X.

2. mw-distances in a quasi-metric space and examples

Definition 1 ([2])**.** An mw-**distance** on a quasi-metric space (X,d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:

(W1) $q(x,y) \leq q(x,z) + q(z,y)$ for all $x, y, z \in X$;

(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;

(mW3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(x,z) \leq \delta$ and $q(y,x) \leq \delta$ then $d(y,z) \leq \varepsilon$.

Note that every quasi-metric d on X is an mw-distance on (X,d).

Definition 2 ([2])**.** A **strong-mw-distance** on a **quasi-metric** space (X,d) is an mw-distance $q : X \times X \to \mathbb{R}^+$ satisfying the following condition:
(mW2) $q(\cdot,x) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$.

It is easy to prove that every strong-mw-distance on a quasi-metric space (X,d) generates a w-distance on this quasi-metric space.

Proposition 3. *Let (X,d) be a quasi-metric space and let q be a strong-mw-distance on (X,d). Then, for all $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$, the function $q_1 : X \times X \to \mathbb{R}^+$ defined by $q_1(x,y) = \alpha q(x,y) + \beta q(y,x)$ is a w-distance on the quasi-metric space (X,d).*

We will show now some examples of mw-distances and strong-mw-distances defined on quasi-metric spaces. These examples are included in [2].

Example 1. Let $X = \mathbb{R}$ and let d_S be the quasi-metric on X given by $d_S(x,y) = y - x$ if $x \leq y$, and $d_S(x,y) = 1$ if $x > y$. The quasi-metric d_S induces the Sorgenfrey topology on $\mathbb{R}$. Since d_S is a quasi-metric, d_S is an mw-distance on the T_1 quasi-metric space (X, d_S). Furthermore, d_S is a strong-mw-distance on (X, d_S).

Example 2. Let $(X, \preceq, \|.\|)$ be a normed lattice. Denote by X^+ the positive cone of X, i.e., $X^+ := \{x \in X : \mathbf{0} \preceq x\}$, and let $\|.\|^+$ be the asymmetric norm on X given by $\|x\|^+ = \|x \vee \mathbf{0}\|$ for all $x \in X$ (see e.g. [8]). The function d defined by $d(x, y) = \|y - x\|^+$ for all $x, y \in X$, is a quasi-metric on X, then (X^+, d_+) is a quasi-metric space, where d_+ denotes the restriction of d to X^+. The function q defined by $q(x, y) = \|y\|$ for all $x, y \in X^+$, is a strong-mw-distance on (X^+, d_+).

Example 3. Consider the quasi-metric space $(\mathbb{R}, d)$ where $d(x, y) = (y - x) \vee 0$. Then $q = d$ is an mw-distance but q is not a strong-mw-distance, because the condition (mW2) does not hold.

3. Partial metrics and mw-distances

In [16] it is studied the relation between mw-distances and partial metrics (quasi-metrics). A partial metric (quasi-metric) is a generalization of the notion of metric (quasi-metric) such that the distance of a point from itself is not necessarily zero. The notion of partial metric was introduced by Matthews [18] as a part of the study of programming language semantics. Later on, Künzi [14] extended this notion to nonsymmetric case.

Definition 4 ([18])**.** A **partial metric** on a set X is a function $p : X \times X \to [0, \infty)$ satisfying:
(1.a) $p(x, x) \leq p(x, y)$;
(1.b) $p(x, x) \leq p(y, x)$;
(2) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$;
(3) $x = y \leftrightarrow p(x, x) = p(x, y)$ and $p(y, y) = p(y, x)$;
(4) $p(x, y) = p(y, x)$.
A **partial metric space** is a pair (X, p) such that X is a set and p is a partial metric on X. The partial metric p induces a T_0 topology τ_p on X which has a base the family of open p-balls $\{B_p(x, \varepsilon) : \ x \in X, \ \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : \ p(x, y) < \varepsilon\}$.

Definition 5 ([14])**.** A **partial quasi-metric** on a set X is a function $p : X \times X \to [0, \infty)$ that verifies the conditions (1.a), (1.b), (2) and (3) of Definition 4. A **partial quasi-metric space** is a pair (X, p) such that X is a set and p is a partial quasi-metric on X. If p satisfies all these conditions except (1b), the function p is called **lopsided partial quasi-metric**.

In a partial (quasi-)metric space (X, p) the partial (quasi-)metric p induces a quasi-metric d_p on X given by $d_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$, and the topology generated by p is the same that the topology generated by d_p. In addition, the function $p^s : X \times X \to [0, \infty)$ given by $p^s(x, y) = d_p(x, y) + d_p(y, x) = p(x, y) + p(y, x) - p(x, x) - p(y, y)$ is a metric on X.

The relationship between mw-distances and partial metrics (quasi-metrics) appears in a natural way as shown in the following result.

Proposition 6. *(a) If (X, p) is a partial metric space then p is both an mw-distance and a w-distance on the quasi-metric space (X, d_p), where $d_p(x, y) = p(x, y) - p(x, x)$.*

(b) If (X, p) is a partial quasi-metric space then p is a mw-distance on the quasi-metric space (X, d_p), where $d_p(x, y) = p(x, y) - p(x, x)$. But p is not necessarily a w-distance on (X, d_p).

4. Results

There are several notions of Cauchy sequence and of complete quasi-metric space in the literature (see e.g. [13]). In this paper we shall use the following general notions.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a quasi-metric space (X, d) is said to be Cauchy if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n_0 \leq n \leq m$. A quasi-metric space (X, d) is called complete if every Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ in the metric space (X, d) converges with respect to the topology $\tau_{d^{-1}}$ (i.e., there exists $z \in X$ such that $d(x_n, z) \to 0$).

Recently, we have obtained a fixed point theorem for generalized contractions with respect to mw-distances on complete quasi-metric spaces. Our approach uses a kind of functions considered by Jachymski in [10].

Theorem 7 (Theorem 2 of [2])**.** *Let f be a self-map of a complete quasi-metric space (X, d). If there exist a strong-mw-distance q on (X, d) and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, and*

$$q(fx, fy) \leq \phi(q(x, y)),$$

for all $x, y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z, z) = 0$.

On the other hand, we have proved in [3] a quasi-metric version of Caristi's fixed point theorem [6] by using mw-distances. Our result generalizes a recent result obtained by Karapinar and Romaguera in [12].

Theorem 8 (Theorem 1 of [3])**.** *Let T be a self mapping of a complete quasi-metric space (X, d) and let q be an mw-distance on (X, d). If there exists a proper bounded below and nearly lower semicontinuous function for $\tau_{d^{-1}}$, $\varphi : X \to \mathbb{R} \cup \{\infty\}$ such that for all $x \in X$:*

$$q(x, Tx) + \varphi(Tx) \leq \varphi(x)$$

then there exists $z \in X$ such that $\varphi(Tz) = \varphi(z)$ and $q(z, Tz) = 0$.

As we mentioned before, our next aim is to obtain fixed point theorems for multivalued maps on quasi-metric spaces with mw-distances. Latif and Al-Mezel [15] extended Mizoguchi-Takahashi's theorem to complete T_1 quasi-metric spaces by using w-distances. Later on, Marín, Romaguera and Tirado [17] generalized this result for multivalued maps.

Theorem 9 (Theorem 1 of [17])**.** *Let $(X, \preceq, d)$ be a complete preordered quasi-metric space and $T : X \to C_d(X)$ be a generalized $w_{\preceq}$-contractive set-valued map. Then T has a fixed point.*

At present, we are trying to prove a result similar to this one by replacing the w-distance with an mw-distance.

References

[1] C. Alegre (2015): *Fixed points theorems and characterizations of metric completeness using mw-distances*, Proceedings of WiAT'15, 5-11, 2015.

[2] C. Alegre, J. Marín (2015): *Modified w-distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces*, CITA 2014, Topology and Its Applications, (to appear) 2015.

[3] C. Alegre, J. Marín (2015): *A Caristi fixed point theorem for complete quasi-metric spaces by using mw-distances*, preprint.

[4] C. Alegre, J. Marín, S. Romaguera (2013): *Fixed points for generalized contractions with respect to w-distances and Meir-Keeler functions,* Proceedings of the Conference in Applied Topology WiAT'13, 53-58.

[5] C. Alegre, J. Marín, S. Romaguera (2014): *A fixed point theorem for generalized contractions involving to w-distances on complete quasi-metric spaces*, Fixed Point Theory and Applications, vol 2014, 2014:40, 1-8.

[6] J. Caristi, W.A. Kirk (1975): *Geometric fixed point theory and and inwardness conditions*, in: The Geometry of Metric and Linear Spaces, Lecture Notes in Mathematics, vol. **490**, Springer-Verlag, Berline, Heidelberg, New York, pp. 74-83.

[7] S. Cobzas (2012): *Functional Analysis in Asymmetric Normed Spaces*, Birkhauser.

[8] J. Ferrer, V. Gregori, C. Alegre (1993): *Quasi-uniform structures in linear lattices*, The Rocky Mountain Journal of Mathematics **23**, 877-884.

[9] P. Fletcher, W.F. Lindgren (1982): *Quasi-Uniform Spaces*, Marcel Dekker, New York.

[10] J. Jachymski (1995): *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl., **194**, 293–303.

[11] O. Kada, T. Suzuki, W.Takahashi (1996): *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Mathematica Japonica **44**, 381-391.

[12] E. Karapinar, S. Romaguera (2015): *On the weak form of Ekeland ś principle in quasi-metric spaces,* Topology and its Applications, **184**, 54-60.

[13] H.P.A. Künzi (2001): *Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology*, in: C.E. Aull, R. Lowen (Eds.), Handbook of the History of General Topology, vol. **3**, Kluwer, Dordrecht, pp. 853-968.

[14] H.P.A. Künzi, H. Pajoohesh, and M. P. Schellekens, *Partial quasi-metric,* Theoretical Computer Science, **365,** 237-246, 2006.

[15] A. Latif, S.A. Al-Mezel (2011): *Fixed point results in quasimetric spaces*, Fixed Point Theory and Applications , Article ID 178306, 8 pages.

[16] J. Marín (2015): *Partial quasi-metrics and mw-distances*, Proceedings of WiAT'15, 89-98, 2015.

[17] J. Marín, S. Romaguera, P. Tirado (2013): *Generalized Contractive Set-Valued Maps on Complete Preordered Quasi-Metric Spaces,* Journal of Functions Spaces and Applications 2013, Article ID 269246, 6 pages.

[18] S.G. Matthews, *Partial metric topology*, Proceedings 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., **728**, 183-197, 1994.

[19] S. Park (2000): *On generalizations of the Ekeland-type variational principles*, Nonlinear Analysis 39, 881-889.

Para seguir leyendo haga click aquí